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THE PROPAGATION OF OSCILLATIONS FROM A POINT SOURCE IN AN ANISOTROPIC PLANE AND HALF-PLANE WITH A THIN COATING†

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Two dynamical problems of the propagation of oscillations in an anisotropic medium generated by a point harmonic force are considered. In the first problem steady harmonic oscillations in an anisotropic plane are investigated. The solution can be reduced to solving a system of second-order elliptic equations for the steady part of the displacements. As the second problem, within the framework of the basic physical model [1], the dynamical contact problem for an anisotropic half-plane reinforced along its boundary by an infinite elastic coating in the form of a thin cover is considered. The generating point force is harmonic. The solution of each of the two problems is constructed by the method of Fourier transforms. Using Lighthill's method [2] and the method of stationary phase [3], asymptotic formulae are obtained for the strains and stresses, in which surface, quasilongitudinal and quasitransverse waves can be distinguished explicitly.

Contact problems of the perturbation of an electroelastic half-plane by a single electrode were considered previously in [4, 5], as was the dynamical contact problem for an isotropic elastic half-plane reinforced by an infinite or semi-infinite thin cover [1].

1. THE PROPAGATION OF OSCILLATIONS FROM A POINT SOURCE IN AN ANISOTROPIC PLANE

Suppose that a point harmonic force $\delta(x)\delta(z)e^{i\omega t}$ directed along the z axis acts at the origin $x = 0$, $z = 0$ of a system of coordinates in an unbounded elastic anisotropic medium. We shall consider solutions of the form

$$u^{(n)}(x,z,t) = u_n(x,z)e^{-i\omega t}, \quad \omega > 0$$

Here and henceforth $n = 1, 3$. The functions $u_n(x, z)$ must satisfy the system of equations

$$\begin{aligned} &c_1\frac{\partial^2 u_1}{\partial x^2}+c_2\frac{\partial^2 u_3}{\partial x\,\partial z}+c_3\frac{\partial^2 u_1}{\partial z^2}+\rho\omega^2 u_1=0 \\ &c_3\frac{\partial^2 u_3}{\partial x^2}+c_2\frac{\partial^2 u_1}{\partial x\,\partial z}+c_4\frac{\partial^2 u_3}{\partial z^2}+\rho\omega^2 u_3+\delta(x)\delta(z)=0 \end{aligned} \tag{1.1}$$

It is assumed that the coefficients $c_1, \ldots, c_4$ are expressed in terms of the elasticity constants of the medium and satisfy the conditions of strong hyperbolicity and positive definiteness of elastic energy [6]

$$-2(\alpha\beta)^{1/2} < \gamma < 1+\alpha\beta \tag{1.2}$$

$$\alpha=\frac{c_3}{c_1}, \quad \beta=\frac{c_3}{c_4}, \quad \gamma=1+\alpha\beta-\frac{c_2^2}{(c_1c_4)}$$

We consider the case

$$0<\alpha<1, \quad 0<\beta<1 \tag{1.3}$$

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By the Fourier transform method with respect to x, we shall construct a solution of (1.1) representing an outgoing wave as $x^2 + z^2 \to \infty$. We obtain the following expressions for the displacements

$$u_1(x,z) = \frac{\operatorname{sign} z}{2\pi} \int_{-\infty}^{\infty} \sum_{m=1}^{2} (-1)^m i\sigma c_2 v_m(\sigma) d\sigma$$
$$u_3(x,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{m=1}^{2} (-1)^m (c_1\sigma^2 - c_1 k_1^2 - c_3\gamma_m) v_m(\sigma)\gamma_1^{-1} d\sigma \tag{1.4}$$

where

$$v_m(\sigma) = [2c_3c_4(\gamma_1 - \gamma_2)]\exp[-i(\sigma x - i\gamma_m(\sigma)|z|)]$$
$$\gamma_m(\sigma) = \left[\frac{Z(\sigma) + (-1)^{m+1}(U(\sigma))^{1/2}}{2\alpha}\right]^{1/2}$$
$$Z(\sigma) = \sigma^2\gamma - k_2^2\alpha(1+\beta), \quad U(\sigma) = Z^2(\sigma) - 4\alpha\beta(\sigma^2 - k_1^2)(\sigma^2 - k_2^2),$$
$$k_1^2 = \rho\omega^2 / c_1, \quad k_2^2 = \rho\omega^2 / c_3$$

Here and henceforth $m = 1, 2$.

We will begin by studying the functions $\gamma_m(\sigma)$. We note that $\pm k_{m+2}$ are branching points of order two of the inner radical in the expressions for $\gamma_m(\sigma)$ and

$$\pm k_{m+2} = \pm\alpha^{1/2} k_2 M_1^{-1/2}[M_2 + (-1)^{m+1} T^{1/2}]^{1/2}$$
$$M_1 = \gamma^2 - 4\alpha\beta, \quad M_2 = \gamma(1+\beta) - 2\beta(1+\alpha)$$
$$M_3 = (1-\beta)^2, \quad T = M_2^2 - M_1 M_3 = 4\alpha c_3^2 c_4^{-2}[(\alpha+\beta) - \gamma]$$

If $Z(\pm k_1) < 0$ and $Z(\pm k_2) > 0$, then $\pm k_1$, $\pm k_2$ will correspondingly be the branching points of order two of the outer radical in the expressions for $\gamma_m(\sigma)$.

The following expansions are therefore valid

$$\gamma_m(\sigma) = a_m^{\pm}(\sigma \pm k_m)^{1/2} + \ldots, \quad (\sigma - k_m)^{1/2} = -i(k_m - \sigma)^{1/2}$$
$$(-\sigma - k_m)^{1/2} = -i(\sigma + k_m)^{1/2} \quad a_m^- = [(-1)^m 2k_m\beta(k_2^2 - k_1^2)/Z(k_m)]^{1/2}$$
$$Z(k_m) = k_m^2\gamma - k_2^2\alpha(1+\beta) \quad a_m^+ = -ia_m^-$$

It follows from (1.2) and (1.3) that these expansions hold only when γ lies in the range $(\alpha(1 + \beta), 1 + \alpha\beta)$, where $\alpha(1 + \beta)$ can be determined from the equation $Z(k_2) = 0$. If $\gamma = \alpha(1 + \beta)$, then $\pm k_2 = \pm k_4$ will be fourth-order branching points for $\gamma_m(\sigma)$ and $\pm k_1$ will still be second-order branching points for $\gamma_1(\sigma)$. Thus

$$\gamma_m(\sigma) = b_m^{\pm}(\sigma \pm k_2)^{1/4} + \ldots, \quad (\sigma - k_2)^{1/4} = e^{-i\pi/4}(k_2 - \sigma)^{1/4}$$
$$(-\sigma - k_2)^{1/4} = e^{-i\pi/4}(k_2 + \sigma)^{1/4}, \quad b_2^- = e^{-i\pi/4} b_1^+$$
$$b_1^- = e^{i\pi/4} b_1^+, \quad b_1^+ = \alpha^{1/4}[4k_2\beta(k_2^2 - k_1^2)]^{1/4}$$

Since $Z(k_2) = 0$ for $\gamma = \alpha(1 + \beta)$, it follows that $Z(k_2) < 0$ and $Z(k_1) < 0$ when $-2(\alpha\beta)^{1/2} < \gamma < \alpha(1 + \beta)$. This means that $\gamma_2(\sigma)$ has no branching points and $\pm k_1 = \pm k_2$ will be second-order branching points for $\gamma_1(\sigma)$. Consequently

$$\gamma_1(\sigma) = d_m^{+}(\sigma \pm k_m)^{1/2} + \ldots, \quad d_m^+ = -id_m^-$$
$$d_m^- = [2k_m\beta(k_1^2 - k_2^2)/(Z(k_m))]^{1/2}$$

If the branching points of $\gamma_m(\sigma)$ are known for various values of γ, asymptotic expressions for $|x| \to \infty$ can be obtained by Lighthill's method. We have
when $\alpha(1+\beta) < \gamma < 1 + \alpha\beta$

$$u^{(3)}(x,0,t) = \frac{A_1(a_2^-)E_2}{\pi^{1/2}|x|^{1/2}} \; \frac{A_3E_2 + ia_1E_1}{2\pi^{1/2}|x|^{3/2}} + o(|x|^{-5/2})$$

for $-2(\alpha\beta)^{1/2} < \gamma < \alpha(1+\beta)$

$$u^{(3)}(x,0,t) = \frac{1}{\pi}\left(\frac{A_1(d_2^-)}{|x|^{1/2}} - \frac{A_3}{|x|^{3/2}}\right)E_2 + \frac{iB_1(d_1^-)E_1}{2\pi^{1/2}|x|^{3/2}} + o(|x|^{-5/2}) \tag{1.5}$$

when $\gamma = \alpha(1+\beta)$

$$u^{(3)}(x,0,t) = \frac{(k_2^2 - k_1^2)^{1/2}\Gamma(1/4)F}{\alpha k_2^2\pi(b_1^- + b_2^-)|x|^{1/4}} + \ldots + \frac{iB_1(a_1^-)E_1}{2\pi^{1/2}|x|^{3/2}} + o(|x|^{-7/4})$$

Here

$$E_m = \exp[-i(\omega t - k_m|x| - \pi/4)]$$

$$F = \exp[-i(\omega t - k_2|x| - 3\pi/4)], \quad u^{(1)}(x,0,t) \equiv 0$$

$$B_1(\lambda) = \frac{\lambda[c_3\lambda^2 - 2k_1c_1(c_1c_4)^{1/2}]}{4k_1c_1c_3(k_2^2 - k_1^2)}$$

$$A_1(\lambda) = \frac{\lambda(c_1c_4)^{1/2}}{4c_3k_2}$$

$$A_3 = \lim_{\sigma\to k_2} 2(\sigma - k_2)^{1/2}\frac{d}{d\sigma}\left[(\sigma-k_2)^{1/2}\frac{d}{d\sigma}(\sigma-k_2)^{1/2}\bar{u}_3(\sigma,0)\right]$$

It is also of interest to obtain asymptotic formulae when $z \neq 0$. In this case it can be assumed without loss of generality that $x > 0$ and $z > 0$.

Let us investigate the critical points of the functions

$$\lambda_m(\alpha_m) = \alpha_m\cos\theta - i\gamma_m(\alpha_m)\sin\theta$$

i.e. the zeros of the functions $\lambda_m^{(i)}$ ($i = 1, 2, \ldots$), where $\lambda_m^{(i)}$ are ith order derivatives of the function $\lambda_m(\alpha_m)$, which must satisfy the equation

$$\operatorname{tg}\theta = (id\gamma_m/(d\alpha_m))^{-1}, \quad \alpha_m = \sigma_m + i\tau_m \tag{1.6}$$

$$\left(i\frac{d\gamma_m}{d\alpha_m} = (-1)^m\frac{\alpha_m[\gamma\gamma_m^2 - \beta(2\alpha_m^{(2)} - k_1^2 - k_2^2)]}{i(U(\alpha_m))^{1/2}\gamma_m}\right)$$

for every θ.

Hence it follows that the equations under consideration will have solutions only if $id\gamma_m/(d\alpha_m)$ are real.

We consider the following cases depending on the values of γ.

1. $\gamma_* < \gamma < 1 + \alpha\beta$. In this case the wave fronts have four lacunae, which lie between the coordinate axes (x, z) [7]. The quantity γ_* is a root of the equation

$$[3(1-\beta)P-(1+\beta)\gamma-2\beta(1+\alpha)][(1-\beta)P+Q]^{1/2}+2[(1+\beta)P-(1-\beta)\gamma][2(1-\beta)P]^{1/2}=0$$
$$P=(\gamma^2-4\alpha\beta)^{1/2},\quad Q=2\beta(1-\alpha)-\gamma(1-\beta)$$

It can be shown that $\lambda_2^{(2)}(\alpha_2)=0$ when $\alpha_2=\alpha'_{20}$ and $\alpha_2=\alpha''_{20}$, where α'_{20} and α''_{20} can be determined from the condition $\gamma_2^2(\alpha_2)=0$. To each value of θ from the interval (θ_*, θ_{**}) there correspond three zeros α_{20}^k $(k=1,2,3)$ of $\lambda_2^{(1)}(\alpha_2)$, which can be determined from (1.6). The end-points of the interval are given by arctg $(id\gamma_2/d\alpha_2)^{-1}$ with $\alpha_2=\alpha'_{20}$ for θ_* and with $\alpha_2=\alpha''_{20}$ for θ_{**}. To each of the remaining values of θ in the interval $(0, \pi/2)$ there corresponds one zero of $\lambda_m^{(1)}(\alpha_m)$.

2. $\gamma=\gamma_*$. The lacunae become a point. Consequently, $\alpha_2=\alpha'''_{20}=\alpha''_{20}=\alpha'_{20}$ will be a zero of $\lambda_2^{(3)}(\alpha_2)$. It can be shown that $\lambda_2^{(4)}(\alpha'''_{20})\neq 0$. The quantity α'''_{20} can be determined from the condition $\gamma_2^3(\alpha_2)$. The critical points of $\gamma_m(\alpha_m)$ for the remaining values of λ from the interval $(-2(\alpha\beta)^{1/2}, \gamma_*)$ can be studied in a similar way.

If the critical points of the functions $\gamma_m(\alpha_m)$ are known, asymptotic expressions for $u^{(n)}(r,\theta,t)$ can be obtained by the method of stationary phase using (1.4) and polar coordinates.

In case 1 we find

for $0<\theta<\theta_*$ and $\theta_*<\theta<\pi/2$

$$u^{(n)}(r,\theta,t)=B_n^{(1)}(\alpha_{10})+B_n^{(2)}(\alpha_{20})+o(r^{-3/2})$$

for $\theta=\theta_*$

$$u^{(n)}(r,\theta,t)=B_n^{(1)}(\alpha_{10})+A_n^{(2)}(\alpha'_{20})F_3(\alpha'_{20})[r|\lambda_2^{(2)}(\alpha'_{20})|]^{-1/3}+o(r^{-2/3}) \tag{1.7}$$

for $\theta_*<\theta<\theta_{**}$

$$u^{(n)}(r,\theta,t)=B_n^{(1)}(\alpha_{10})+\sum_{k=1}^{3}B_n^{(2)}(\alpha_{20}^{(k)})+o(r^{-3/2})$$

and $\theta=\theta_{**}$

$$u^{(n)}(r,\theta,t)=B_n^{(1)}(\alpha_{10})+A_n^{(2)}(\alpha''_{20})F_3(\alpha''_{20})[r|\lambda_2^{(2)}(\alpha''_{20})|]^{-1/3}+o(r^{-2/3})$$

In case 2 the lacunae becomes a point and
for $0<\theta\neq\theta_*<\pi/2$

$$u^{(n)}(r,\theta,t)=B_n^{(1)}(\alpha_{10})+B_n^{(2)}(\alpha_{20})+o(r^{-3/2})$$

and for $\theta=\theta_*$

$$u^{(n)}(r,\theta,t)=B_n^{(1)}(\alpha_{10})+A_n^{(2)}(\alpha'''_{20})F_4(\alpha'''_{20})[r\lambda_2^{(4)}(\alpha'''_{20})]^{-1/4}+o(r^{-3/4})$$

Here

$$F_m(\alpha_{m0})=\exp[-i(\omega t+\lambda_m(\alpha_{m0})r+\tfrac{1}{4}\pi\,\mathrm{sign}\,\lambda_m^{(2)}(\alpha_{m0}))]$$
$$F_3(\alpha'_{20})=\Gamma(1+\tfrac{1}{3})3^{5/6}2^{1/3}\exp[-i(\omega t+\lambda_2(\alpha'_{20})r)]$$
$$F_4(\alpha'''_{20})=\Gamma(1+\tfrac{1}{4})2^{7/4}3^{1/4}\exp[-i(\omega t+\lambda_2(\alpha'''_{20})r+\tfrac{1}{8}\pi\,\mathrm{sign}\,\lambda_2^{(4)}(\alpha'''_{20})]$$
$$\lambda_m(\alpha_m)=\alpha_m\cos\theta-i\gamma_m(\alpha_m)\sin\theta$$

$$\alpha_m=\sigma_m+i\tau_m$$
$$A_1^{(m)}(\lambda)=(-1)^m ic_2\lambda F_5(\lambda)$$
$$A_3^{(m)}(\lambda)=(-1)^m[c_1\lambda^2-c_3k_1^2-c_3\gamma_m^{(2)}(\lambda)]\,\gamma_m^{-1}(\lambda)F_5(\lambda)$$
$$F_5^{-1}(\lambda)=2(2\pi)^{1/2}c_3c_4[\gamma_1^2(\lambda)-\gamma_2^2(\lambda)]$$
$$B_n^{(m)}(\lambda)=A_n^{(m)}(\lambda)F_m(\lambda)[r\lambda_m^{(2)}(\lambda)]^{-1/2}$$

The series obtained above enables us to determine the amplitudes of quasilongitudinal and quasi-transverse waves in a region far from the source. The first term in each of the series (1.5) and (1.7) represents quasilongitudinal waves, the second term representing quasitransverse ones. Note that the quasitransverse waves corresponding to cusp points on the wave front decay as $r^{-1/3}$, i.e. slower than the waves related to ordinary points. When the lacunae become points, the quasitransverse waves decay as $r^{-1/4}$.

2. THE PROPAGATION OF OSCILLATIONS FROM A POINT SOURCE IN AN ANISOTROPIC HALF-PLANE WITH A THIN COATING

We shall solve an auxiliary problem of the steady oscillations of an anisotropic half-plane.

Suppose that a horizontal unit harmonic force $\delta(x)e^{-i\omega t}$ with frequency ω concentrated at the origin of the system of coordinates acts on the boundary of an anisotropic half-plate.

The steady parts of the displacements satisfy the homogeneous system of equations (1.1) with boundary conditions

$$(c_2-c_3)\frac{\partial u_1}{\partial x}+c_4\frac{\partial u_3}{\partial z}=0,\quad c_3\left(\frac{\partial u_1}{\partial z}+\frac{\partial u_3}{\partial x}\right)=-\delta(x)\quad (z=0) \tag{2.1}$$

By the Fourier transform method with respect to x, we shall construct a solution of the homogeneous equations (1.1) with boundary conditions (2.1), representing an outgoing wave as $x^2+z^2\to\infty$. For the displacements we obtain

$$\begin{aligned}
&u^{(n)}(x,0,t)=\frac{e^{-i\omega t}}{2\pi}\int\limits_{-\infty}^{\infty} w_n(\sigma)\kappa(\sigma)d\sigma\\
&w_1(\sigma)=c_4(\sigma^2-k_2^2)^{1/2}(\gamma_1-\gamma_2),\quad \kappa(\sigma)=e^{-i\sigma x}/R(\sigma)\\
&w_3(\sigma)=i\sigma\{[c_1c_4(\sigma^2-k_1^2)]^{1/2}-(c_2-c_3)(\sigma^2-k_2^2)^{1/2}\}\\
&R(\sigma)=\{\sigma^2[c_1c_4-(c_2-c_3)^2]-c_1c_4k_1^2\}(\sigma^2-k_2^2)^{1/2}-c_3k_2^2[c_1c_4(\sigma^2-k_1^2)]^{1/2}
\end{aligned} \tag{2.2}$$

Suppose the anisotropic half-plane is reinforced by an elastic coating taking the form of a fairly thin cover of constant thickness h (see Fig. 1).

The problem consists of finding the distribution of contact stresses along the interface between the coating and the half-plane, given that a horizontal harmonic force $p\delta(x)\sin(\omega t)$ acts on the upper face of the coating. To simplify the computations, we henceforth take a force of the form $p\delta(x)e^{-i\omega t}$. The imaginary part of the solution taken with the opposite sign will obviously be the desired solution.

The following assumptions are made regarding the coating.

The thickness of the coating is many times less than the wavelength of the wave propagating in it. As in [1], we shall assume that because h is small, the flexural stiffness of the coating is negligibly small, so that the normal pressure of the coating on the half-plane can be neglected. In other words, we shall assume that only the shear contact stress acts under the coating, i.e. the system is in a state of uniaxial stress.

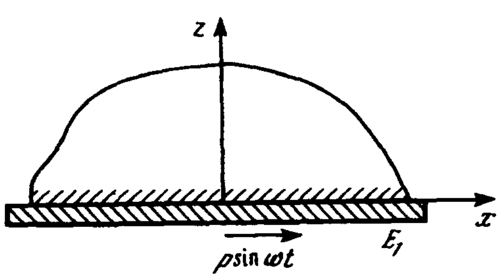

Fig. 1.

In this case the steady oscillations of the coating can be described by the equation [1]

$$d^2u_* / dx^2 + p_1^2 u_* = (E_1 h)^{-1}[\tau_*(x) - p\delta(x)] \tag{2.3}$$
$$p_1^2 = \omega^2 / c^2, \quad c = E_1^{1/2} / \rho_1^{1/2}$$

where c is the velocity of propagation of waves in the coating, $\tau_*(x)$ is the steady part of the unknown shear contact stresses, E_1 and p_1 being, respectively, the modulus of elasticity and the density of the material of the coating.

On the other hand, the amplitude $u_1(x)$ of horizontal displacements of the points on the boundary of the anisotropic half-plane as a function of the stress amplitudes $\tau_*(x)$ applied to the boundary of the anisotropic half-plane is, by (2.2) and the superposition principle, given by the formula

$$u_1(x) = \int_{-\infty}^{\infty} K(|x - s|)\tau_*(s)ds \quad (-\infty < x < \infty)$$

$$K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} w_1(\sigma)\kappa(\sigma)d\sigma$$

Note that the condition

$$u_*(x) = u_1(x) \quad (-\infty < x < \infty)$$

must be satisfied along the interface between the coating and the half-plane. Combined with (2.3), this condition reduces the problem of determining the contact stress amplitude to solving the integro-differential equation

$$\left(\frac{d}{dx^2}^2 + p_1^2\right)\int_{-\infty}^{\infty} K(|x - s|)\tau_*(s)ds = \lambda^*\tau_*(x) - \lambda^* p\delta(x), \quad \lambda^* = (c_4 E_1 h)^{-1} \tag{2.4}$$

The solution of the contact problem of steady oscillations of an anisotropic elastic half-plane reinforced along its boundary by an infinite thin elastic coating can therefore be reduced to the integro-differential equation (2.4).

Applying the Fourier transform to both sides of (2.4) and using a well-known property of convolution, we obtain

$$\tau_*(\sigma) = \lambda^* pR(\sigma)[f(\sigma)]^{-1}$$

Here

$$f(\sigma) = (\sigma^2 - p_1^2)(\sigma^2 - k_2^2)(\gamma_1 + \gamma_2) + \lambda^* R(\sigma) \tag{2.5}$$

Let us investigate the roots of $f(\sigma)$ for various values of γ.

At any point σ on the real axis the functions $\gamma_m(\sigma)$ can take real, pure imaginary, or complex values depending on γ.

If $\gamma < 2(\alpha\beta)^{1/2}$ and $\gamma > \alpha(1 + \beta)$, then $\gamma_1(\sigma)$ is pure imaginary when $|\sigma| < k_2$ and real when $|\sigma| > k_2$. The function $\gamma_2(\sigma)$ is pure imaginary when $|\sigma| < k_2$ and real when $|\sigma| > k_2$, i.e. $\gamma_m(\sigma)$ take only real or pure imaginary values.

If $\gamma < 2(\alpha\beta)^{1/2}$ and $\gamma > \alpha(1 + \beta)$, then $\gamma_1(\sigma)$ is pure imaginary when $|\sigma| < k_1$, real when $k_1 < |\sigma| < \sigma_*$, and complex when $|\sigma| > \sigma_*$. From the continuity of the real part of $\gamma_1(\sigma)$ for $|\sigma| = \sigma_*$ it follows that $\operatorname{Im}\gamma_1(\sigma) = 0$ for $|\sigma| = \sigma_*$. The function $\gamma_2(\sigma)$ is pure imaginary when $|\sigma| < k_2$, real when $k_2 < |\sigma| < \sigma_*$, and complex when $|\sigma| > \sigma_*$; $\operatorname{Im}\gamma_2(\sigma) = 0$ for $|\sigma| = \sigma_*$. The roots of the inner radical of $\gamma_m(\sigma)$ lie on the imaginary and real axes (two lie on the imaginary axis and two, $\sigma = \pm\sigma_*$, lie on the real axis symmetrically about the origin).

If $\gamma = \alpha(1 + \beta)$ and $\gamma < 2(\alpha\beta)^{1/2}$, the functions $\gamma_m(\sigma)$ are complex in the whole interval $|\sigma| > k_2$ and $\pm\sigma_* = \pm k_2$.

If $\gamma = \alpha(1 + \beta)$, $\gamma > 2(\alpha\beta)^{1/2}$ the functions $\gamma_m(\sigma)$ are real in the whole interval $|\sigma| > k_2$ and $\pm\sigma_* = \pm k_2$.

If $\gamma < \alpha(1 + \beta)$ and $\gamma < 2(\alpha\beta)^{1/2}$, then $\gamma_1(\sigma)$ is pure imaginary when $|\sigma| < k_1$, real when $k_1 < |\sigma| < k_2$, pure imaginary when $k_2 < |\sigma| \leqslant \sigma_*$, and complex when $|\sigma| > \sigma_*$, $\operatorname{Re}\gamma_1(\sigma) = 0$ for $\sigma = \sigma_*$. The function $\gamma_2(\sigma)$ is pure imaginary when $|\sigma| \leqslant \sigma_*$ and complex when $|\sigma| > \sigma_*$, $\operatorname{Re}\gamma_2(\sigma) = 0$ for $|\sigma| = \sigma_*$.

Two roots of the inner radical are real, $\sigma = \pm\sigma_*$, and two are pure imaginary. It can be shown that $k_2 < \sigma_* < k_2[\alpha(1 + \beta)/\gamma]^{1/2}$ if $\gamma < \alpha(1 + \beta)$.

On the basis of the above, one can claim that $\gamma_1(\sigma) + \gamma_2(\sigma)$ takes a real value for $\alpha(1 + \beta) \leqslant \gamma < 1 + \alpha\beta$, $|\sigma| \geqslant k_2$ and $-2(\alpha\beta)^{1/2} < \gamma < \alpha(1 + \beta)$, $|\sigma| \geqslant \sigma_*$.

Setting $\sigma^2 = l^2y^2$ $(l^2 = \rho\omega^2)$ in (2.5), after elementary calculations we obtain

$$f(y) = l^2\{l^2(y^2 - p_1^*)(y^2 - c_3^{-1})^{1/2}(\gamma_1(y) + \gamma_2(y)) + \lambda^* R(y)\}$$

$$R(y) = \{[c_1c_4 - (c_2 - c_3)^2]y^2 - c_4\} - (y^2 - c_3^{-1})^{1/2} - (c_1c_4)^{1/2}(y^2 - c_1^{-1})^{1/2}, \quad p_1^{*2} = (\rho c^2)^{-1}$$

Note that in the case of a transversely isotropic medium $R(y)$ has two real roots $y = \pm y_R$ and $y_R > c_3^{-1/2}$ [8].

We consider the following cases.

1. $\alpha(1 + \beta) \leqslant \gamma < 1 + \alpha\beta$. In this case the following three combinations can occur

$$c_3^{-1/2} < p_1^* < y_R; \quad p_1^* < c_3^{-1/2} < y_R; \quad c_3^{-1/2} < y_R < p_1^* \tag{2.6}$$

Let us consider the first case

$$f(\pm p_1^*) = \rho^{3/2}\omega^3\lambda^* R(\pm p_1^*) < 0$$

$$f(\pm y_R) = \rho^2\omega^5(y_R^2 - p_1^*)(y_R^2 - c_3^{-1})^{1/2}[\gamma_1(\pm y_R) + \gamma_2(\pm y_R)] > 0$$

The function $f(y)$ has two real roots, which lie in the intervals $(\pm p_1^*, \pm y_R)$.

Second case. Here

$$f(\pm c_3^{-1/2}) = -\rho^{3/2}\omega^3\lambda^*(c_1c_4)^{1/2}(c_3^{-1} - c_1^{-1})^{1/2} < 0$$

$$f(\pm y_R) = \rho^{3/2}\omega^5(y_R^2 - p_1^*)(y_R^2 - c_3^{-1})^{1/2}[\gamma_1(\pm y_R) + \gamma_2(\pm y_R)] < 0$$

These roots lie in the intervals $(\pm c_3^{-1/2}, \pm y_R)$.

Third case. We have

$$f(\pm y_R) = \rho^{5/2}\omega^5(y_R^2 - p_1^{*2})(y_R^2 - c_3^{-1})^{1/2}[\gamma_1(\pm y_R) + \gamma_2(\pm y_R)] < 0$$

$$f(\pm p_1^*) = \rho^{3/2}\omega^3 R(\pm p_1^*)\lambda^* > 0$$

We denote these roots by $\pm y_R^*$. In the third case $\pm y_R^* \in (\pm y_R, \pm p_1^*)$.

2. $-2(\alpha\,\beta)^{1/2} < \gamma < \alpha(1 + \beta)$. Then we have

$$\begin{aligned}
&(a)\ y_* < y_R < p_1^*, \quad \pm y_R^* \in (\pm y_R, \pm p_1^*)\\
&(b)\ y_* < p_1^* < y_R, \quad \pm y_R^* \in (\pm p_1^*, \pm y_R)\\
&(c)\ p_1^* < y_* < y_R, \quad \pm y_R^* \in (\pm y_*, \pm y_R)\\
&(d)\ p_1^* < y_R < y_*, \quad (e)\ y_R < p_1^* < y_*, \quad (f)\ y_R < y_* < p_1^*
\end{aligned} \tag{2.7}$$

In cases (d)–(f) $f(y)$ has no roots, y_* being the branching points of the inner radical.

It follows that in cases 1 and 2 the roots of $f(y)$ lie in the intervals $(\pm p_1^*, \pm y_R)$ when $y_R > y_*$, $\lambda^* < \rho y_R^2(\rho_1 c_4 h)^{-1}$. When $\lambda^* = \rho y^2 R(\rho c_4 h)^{-1}$ or $y_R = y_*$, the roots of $f(y)$ are identical with $\pm y_R$. If $y_R > y_*$, $\lambda^* > \rho y^2 R(\rho_1 c_4 h)^{-1}$, the roots belong to the intervals $(\pm y_R, p_1^*)$. If $y_R < y_*$, then $f(y)$ has no roots.

Now we shall derive an asymptotic formula for the tangential contact stress

for $\alpha(1 + \beta) < \gamma < 1 + \alpha\beta$

$$-\operatorname{Im}\tau(x,t) = T_R\Omega + \frac{T(a_1^-)E_{11}(a_1^-)}{2\pi^{1/2}|x|^{3/2}} + \frac{\Pi(a_2^-)\operatorname{Re}E_2}{2\pi^{1/2}|x|^{3/2}} + o(|x|^{-5/2}) \tag{2.8}$$

for $\gamma = \alpha(1 + \beta)$

$$-\operatorname{Im}\tau(x,t) = T_R\Omega + \frac{T(a_1^-)E_{11}(a_1^-)}{2\pi^{1/2}|x|^{3/2}} + \frac{3\Gamma(3/4)\Pi_* E}{2^{5/2}\pi|x|^{7/4}} + o(|x|^{-9/4})$$

for $\alpha(1 + \beta) < \gamma < 1 + \alpha\beta$

$$-\operatorname{Im}\tau(x,t)=T_R\Omega+\frac{T(d_1^-)E_{11}(d_1^-)}{2\pi^{1/2}|x|^{3/2}}-\frac{\Pi(d_2^-)\operatorname{Re}E_2}{2\pi^{1/2}|x|^{3/2}}+o(|x|^{-5/2})$$

Here

$$\Omega=\cos(\omega t-\sigma_R^*|x|),\quad E=\cos(\omega t-k_2|x|-\pi/8)$$

$$E_{11}(\lambda)=\cos[\omega t-k_1|x|-\pi/4+\operatorname{arctg}(T_{**}(\lambda)/T_*(\lambda))]$$

$$T(\lambda)=[T_*^2(\lambda)+T_{**}^2(\lambda)]^{1/2}$$

$$T_R=\Omega_1^{-1}(\sigma_R^*)R(\sigma_R^*)$$

$$T_*(\lambda)=(k_1^2-k^2)k_1^2\lambda\varepsilon(\lambda)\Omega_-(\lambda)\Omega_+^{-2}(\lambda)$$

$$T_{**}(\lambda)=2(2k_2\beta)^{1/2}k_1^4\lambda^*\lambda^2(k_1^2-k^2)L\varepsilon(\lambda)\Omega_+^{-2}(\lambda)$$

$$\varepsilon(\lambda)=L\lambda^2-2c_1^2k_1^2,\quad L=(c_2-c_3)^2-c_1c_4$$

$$\Omega_1(\sigma_R^*)=d/(d\sigma)[f(\sigma)]_{\sigma=\sigma_R^*}$$

$$\Omega_\pm(\lambda)=2(k_1^2-k^2)^2(k_2^2-k_1^2)k_2\beta\pm\lambda^{*2}\lambda^2k_1^4L$$

$$T_{**}(\lambda)/T_*(\lambda)=2(2k_2\beta)^{1/2}\lambda^*\lambda(k_1^2-k^2)L\Omega_-^{-1}(\lambda)$$

$$\Pi(\lambda)=\frac{2(k^2-k_2^2)\beta^{1/2}}{\lambda^{*2}\lambda(c_3c_4)^{1/2}c_3k_2}$$

$$\Pi_*(\lambda)=\frac{(k^2-k_2^2)b_1^+}{3\lambda^{*2}c_3(c_1c_4)^{1/2}k_2^{3/2}(k_2^2-k_1^2)^{1/2}}$$

The first term in each of the series (2.8) represents the stress caused by surface waves propagating with velocity $\upsilon_R^*=\omega/\sigma_R^*$. The second term is due to quasilongitudinal waves, while the third one comes from quasitransverse waves in a region far away from the source.

Let us compare the velocities of propagation of surface stress waves in an anisotropic half-plane with a thin coating and in an anisotropic half-plane without a coating. We have a non-uniform half-plane in the former case and a uniform one in the latter.

If a horizontal concentrated harmonic force is applied at the boundary of the uniform anisotropic half-plane, the velocity of propagation of surface waves, which depends only on the elastic constants of the material of the anisotropic half-plane, is given by $\upsilon_R=\omega/\sigma_R^*$.

Now let the same force be applied at the boundary of the non-uniform half-plane. Considering separately the oscillations of the coating within the framework of the assumptions adopted above, we find that the waves in the coating propagate with velocity $(E_1/\rho_1)^{1/2}$.

If $\sigma_R>\sigma_*$ and $E_1/\rho_1>\omega^2/\sigma_R^2$, the roots of $f(y)$ lie in the intervals $(\pm p_1^*,\ \pm\sigma_R/\omega)$.

On the other hand, since the velocity of propagation of surface waves in the non-uniform half-plane equals $\upsilon_R^*=\omega/\sigma_R^*$, where σ_R^*/ω is a root of $f(y)$, it follows that the velocity of propagation of surface waves is greater than that of free waves in the uniform half-plane.

Now let $\sigma_R>\sigma_*$ and $E_1/\rho_1<\omega^2/\sigma_R^2$. Then the roots of the function belong to the intervals $(\pm\sigma_R/\omega,\ \pm p_1^*)$. This means that the velocities of propagation of surface waves are less than those of similar waves in the uniform half-plane.

When $\sigma_R=\sigma_*$ or $E_1/\rho_1=\omega^2/\sigma_R^2$, the roots of $f(y)$ are the same as those of $R(y)$. As a result, $\tau(\sigma)$ has no poles on the real axis. No surface waves arise in these cases. Finally, when $\sigma_R<\sigma_*$ the function $f(y)$ has no real roots, so that no surface waves arise either.

Note that there is a phase shift between the quasilongitudinal waves. From (2.8) it follows that the phase shift is negative when $k_1<k$ and $\Omega_-^{-1}(\lambda)>0$ and positive when $k_1<k$, $\Omega_-^{-1}(\lambda)<0$ and $k_1>k$, $\Omega_-^{-1}(\lambda)>0$. This means that in the first case the quasilongitudinal waves lag behind those in the anisotropic half-plane without a coating, while they lead in the other case. As can be seen from (2.8), the stress due to quasilongitudinal waves can be neglected for large x when $k_1=k$. There are no phase shifts for quasitransverse waves. This is consistent with the adopted model of the coating.

The following conclusions can be drawn from the above results.

1. If the lacuna does not intersect the x axis, i.e. $\alpha(1 + \beta) < \gamma < 1 + \alpha\beta$, surface waves will arise in the non-uniform anisotropic half-plane. Note that a study was carried out previously [1] for the part of the domain $(\alpha(1 + \beta), 1 + \alpha\beta)$ [6] in which surface waves also arise.

2. When a lacuna occurs on the axis, i.e. $-2(\alpha\beta)^{1/2} < \gamma < \alpha(1 + \alpha\beta)$ and $\sigma_R > \sigma_*$ or $\upsilon_R < \omega/\sigma_*$, surface waves arise again in the non-uniform anisotropic half-plane.

Surface waves do not arise when $\upsilon_R \geqslant \omega/\sigma_*$. But then the adopted model of the coating is unsuitable, so that the absence of surface waves in the case in question may possibly be explained by the unsuitability of the model of the coating.

Note also that the quasitransverse waves decay as $|x|^{-7/4}$ when the lacunae become points on the x axis, i.e. $\gamma = \alpha(1 + \beta)$.

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